THE INTRINSIC TORSION OF SU(3) AND G_2 STRUCTURES

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To Antonio Naveira on the occasion of his 60th birthday

We analyse the relationship between the components of the intrinsic torsion of an $SU(3)$-structure on a 6-manifold and a G_2-structure on a 7-manifold. Various examples illustrate the type of $SU(3)$-structure that can arise as a reduction of a metric with holonomy G_2.

Introduction

Let G be a subgroup of $SO(n)$. A G-structure on a smooth manifold M of dimension n induces a Riemannian metric g on M. The failure of the holonomy group of the Levi-Civita connection of g to reduce to G is measured by the so-called intrinsic torsion τ. It is known[1,2] that the latter is a tensor which takes values at each point in $T^* \otimes \mathfrak{g}^\perp$ where T^* is the cotangent space and $\mathfrak{g}^\perp$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{so}(n) \cong \bigwedge^2 T^*$.

This note is concerned with the cases

$$\begin{aligned} &(1) \quad SU(3) \subset SO(6), \\ &(2) \quad G_2 \subset SO(7). \end{aligned}$$

The respective tensors τ_1 and τ_2 belong to spaces of dimension 42 and 49. The fact that $SU(3)$ is a maximal subgroup of G_2 gives a direct relationship between the two structures. Indeed, the sets of reductions (1) and (2) are both parametrized by the projective space

$$\mathbb{RP}^7 = \frac{SO(6)}{SU(3)} = \frac{SO(7)}{G_2}. \tag{1}$$

The fact that this space itself admits homogeneous G_2-structures has applications to the study of families of G_2-structures. Moreover, the fibration $\mathbb{RP}^7 \to \mathbb{CP}^3$ is indicative of the way in which G_2-structures can in general be built from almost-Hermitian structures on a 6-manifold.

We begin by describing the tensor τ_1 determined by an $SU(3)$-structure on a 6-manifold M, thereby refining the theory for $U(3)$. An additional summand in the $SU(3)$ case can be used to construct a new conformally invariant torsion tensor. It is well known that a holonomy reduction to $SU(3)$ is characterized by the existence of a symplectic form together with a closed form of 'type $(3,0)$', and it follows that all the components of τ_1 can be calculated in terms of *exterior* derivatives of the forms defining the reduction. The special relevance of 3-forms in describing 6-dimensional structures is already documented,[3] and this paper presents same additional applications.

In the general set-up, the $SU(3)$ reduction leads to a splitting of the Nijenhuis tensor in two equal parts, which give rise to different components of the tensor τ_2 on a 7-manifold $\mathbb{M}$ whose structure reduces from G_2 to $SU(3)$. This aspect of the theory is reminiscent of self-duality in four dimensions, and the G_2 examples analysed in subsequent sections can by analogy be divided into those of self-dual and anti-self-dual type. The distinction arises from whether the 7-manifold is foliated by leaves of dimension 1 or 6.

Our first descriptions of τ_1, τ_2 are 'static' in the sense that they relate to a fixed G-structure and are purely algebraic. We subsequently examine how the components of τ_1 determine those of τ_2 in various situations in which the geometry of the 6- and 7-manifolds are interrelated, with the inclusion $SU(3) \subset G_2$ varying from point to point. The evolution equations discussed by Hitchin[4] are interpreted using the notion of a half-flat $SU(3)$ structure. We provide additional examples of incomplete metrics with holonomy G_2 of the type discovered by Gibbons et al[5] that suggest that half-flat structures occur naturally on 6-dimensional nilmanifolds.

The final section undertakes an investigation of certain cases in which the G_2-manifold $\mathbb{M}$ is a circle bundle over a 6-manifold M endowed with an appropriate structure. We provide an explicit description of τ_2 as a function of τ_1 and a curvature 2-form, and consider the case of the canonical circle bundle over a Kähler 3-fold. When the holonomy of $\mathbb{M}$ reduces to G_2, the quotient $M = \mathbb{M}/S^1$ is a symplectic manifold with a type of generalized Calabi-Yau geometry that we describe briefly in terms of τ_1. Examples of such quotients of the known complete metrics with G_2 holonomy incorporate interesting global features,[6,7] and it is hoped that the techniques of this paper will aid a fuller investigation of this situation.

1 Static SU(3) structures

Let M be a 6-manifold with a $U(3)$-structure. Thus M is equipped with a Riemannian metric g, an orthogonal almost-complex structure J and an associated 2-form ω. The exterior forms on M may be decomposed into types relative to J, and we adopt the following notation[8] at each point:

$$\begin{aligned} T^* &= [\![\Lambda^{1,0}]\!], \\ \bigwedge^2 T^* &= [\![\Lambda^{2,0}]\!] \oplus [\Lambda^{1,1}] \ \cong [\![\Lambda^{2,0}]\!] \oplus [\Lambda_0^{1,1}] \oplus \mathbb{R}, \\ \bigwedge^3 T^* &= [\![\Lambda^{3,0}]\!] \oplus [\![\Lambda^{2,1}]\!] \ \cong [\![\Lambda^{3,0}]\!] \oplus [\![\Lambda_0^{2,1}]\!] \oplus [\![\Lambda^{1,0}]\!], \\ \bigwedge^4 T^* &= [\![\Lambda^{3,1}]\!] \oplus [\Lambda^{2,2}] \ \cong [\![\Lambda^{2,0}]\!] \oplus [\Lambda_0^{1,1}] \oplus \mathbb{R}. \end{aligned} \tag{2}$$

The induced metric distinguishes the circle B consisting of elements of unit norm in the 2-dimensional space $[\![\Lambda^{3,0}]\!]$. An $SU(3)$-structure is determined by the choice of a real 3-form ψ_+ lying in B at each point, or equivalently a section of the associated S^1-bundle $\mathbb{B}$. The associated $(3,0)$-form is

$$\Psi = 2(\psi_+)^{3,0} = \psi_+ + i\psi_- \tag{3}$$

with $\psi_- = J\psi_+$. We may then write

$$\mathbb{B} = \{a\psi_+ + b\psi_- : a^2 + b^2 = 1\}, \tag{4}$$

and this description remains valid locally even if a global reduction from $U(3)$ to $SU(3)$ does not exist.

To be more explicit, we may choose a local orthonormal basis $(e^1, \dots, e^6)$ of T^* such that

$$\Psi = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6).$$

Consequently

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246}, \end{aligned} \tag{5}$$

where e^{135} stands for $e^1 \wedge e^3 \wedge e^5$ etc. These forms are subject to the compatibility relations

$$\begin{aligned} \omega \wedge \psi_\pm &= 0, \\ \psi_+ \wedge \psi_- &= \tfrac{2}{3}\omega^3. \end{aligned} \tag{6}$$

Table 1. $SU(3)$ torsion

component	$\dim_{\mathbb{R}}$	$U(3)$-module	$SU(3)$-module	
$\mathcal{W}_1$	2	$[\![\Lambda^{3,0}]\!]$	$\mathbb{R}$	$\mathbb{R}$
$\mathcal{W}_2$	16	$[\![V]\!]$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)$
$\mathcal{W}_3$	12	$[\![\Lambda_0^{2,1}]\!]$	$[\![S^{2,0}]\!]$	
$\mathcal{W}_4$	6	T	T	
$\mathcal{W}_5$	6	T	T	

The intrinsic torsion of the $U(3)$-structure can be identified with ∇J or $\nabla\omega$ and belongs to the space

$$T^* \otimes \mathfrak{u}(3)^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4,$$

whose four components were first described by Gray–Hervella.[9] The intrinsic torsion τ_1 of the $SU(3)$-structure lies in the enlarged space

$$T^* \otimes \mathfrak{su}(3)^\perp \cong T^* \otimes ([\![\Lambda^{2,0}]\!] \oplus \mathbb{R}) \cong T \otimes (T \oplus \mathbb{R}),$$

given that now $\Lambda^{2,0} \cong \Lambda^{0,1}$. Thus

$$\tau_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

with $\mathcal{W}_5 \cong T$. Properties of the various components are indicated by Table 1.

We denote the component of τ_1 in $\mathcal{W}_i$ in a formal way by W_i, though we shall need to supply more precise definitions shortly. It is well known that the components of ∇J in $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ are determined by $d\omega$, and those in $\mathcal{W}_1 \oplus \mathcal{W}_2$ by the Nijenhuis tensor. The choice of basis (ψ_+, ψ_-) of $[\![\Lambda^{3,0}]\!]$ at each point provides an isomorphism

$$\mathcal{W}_1 \oplus \mathcal{W}_2 = [\![\Lambda^{2,0} \otimes \Lambda^{1,0}]\!] = [\![\Lambda^{3,0} \otimes \Lambda^{1,1}]\!] \cong \mathbb{R}^2 \otimes \mathfrak{u}(3).$$

The rank of $W_1 + W_2$ (or of W_2 on its own) in this tensor product is equal to one of 0,1,2.

Remark. If M is an almost-Hermitian manifold for which the bundle $\mathbb{B}$ is trivial but not actually trivialized, the basis (ψ_+, ψ_-) is defined up to an overall constant action by S^1, and the rank of $W_1 + W_2$ is a global invariant. This situation occurs naturally on 6-dimensional Lie groups of the type considered below.

On a complex manifold, $d\psi_+$ belongs to $\Lambda^{3,1} \oplus \Lambda^{1,3}$ at each point. It follows that the component of $d\psi_+$ in $\Lambda^{2,2}$ is determined by the Nijenhuis tensor, and therefore by $W_1 \oplus W_2$. We may define the two scalar components $W_1^{\pm} \in \mathbb{R}$ of $\mathcal{W}_1$ by

$$\begin{aligned} d\psi_+ \wedge \omega = \psi_+ \wedge d\omega = W_1^+ \,\omega^3, \\ d\psi_- \wedge \omega = \psi_- \wedge d\omega = W_1^- \,\omega^3, \end{aligned}$$

where $\omega^3 = \omega \wedge \omega \wedge \omega$. Similarly, $W_2 = W_2^+ + W_2^-$ in which

$$\begin{aligned} (d\psi_+)^{2,2} = W_1^+ \omega^2 + W_2^+ \wedge \omega, \\ (d\psi_-)^{2,2} = W_1^- \omega^2 + W_2^- \wedge \omega, \end{aligned}$$

so that $W_2^{\pm} \in [\Lambda_0^{1,1}]$ are effective $(1,1)$-forms.

Given that

$$d\psi_+ - i d\psi_- = d\overline{\Psi} \in \Lambda^{1,3} \oplus \Lambda^{2,2},$$

the remaining components of $d\psi_+, d\psi_-$ are related by

$$(d\psi_+)^{3,1} = i(d\psi_-)^{3,1}. \tag{7}$$

It is now clear that the W_5-component of τ_1 arises from (7). In summary, we have

1.1 Theorem The five components of τ_1 are determined by $d\omega$, $d\psi_+$, $d\psi_-$, in the following manner:

$$\begin{aligned} W_1 &\longleftrightarrow (d\omega)^{3,0} \\ W_2 &\longleftrightarrow ((d\psi_+)_0^{1,1},\ (d\psi_-)_0^{1,1}) \\ W_3 &\longleftrightarrow (d\omega)_0^{2,1} \\ W_4 &\longleftrightarrow \omega \wedge d\omega \\ W_5 &\longleftrightarrow (d\psi_{\pm})^{3,1} \end{aligned}$$

(refer to (2)).

It is significant that W_4 and W_5 arise from isotypic summands of the space $T^* \otimes \mathfrak{su}(3)^{\perp}$. Before moving on to seven dimensions, it is convenient to give a more precise definition of these components too in order that they may be compared directly. We shall do this by means of the contraction

$$\lrcorner \ : \bigwedge^k T^* \otimes \bigwedge^n T^* \to \bigwedge^{n-k} T^*$$

that exploits the underlying Riemannian metric, with the convention that $e^{12} \lrcorner\, e^{12345} = e^{345}$ etc.

1.2 Definition The components of τ_1 in $\mathcal{W}_4, \mathcal{W}_5$ are given by

$$\begin{aligned} W_4 &= \tfrac{1}{2}\omega \lrcorner\, d\omega, \\ W_5 &= \tfrac{1}{2}\psi_+ \lrcorner\, d\psi_+. \end{aligned}$$

The coefficient of one half is added with the following examples in mind. Given ω as in (5), suppose that $d\omega = \omega \wedge e^1$ and $d\psi_+ = \psi_+ \wedge e^1$. Then

$$\begin{aligned} W_4 &= \tfrac{1}{2}\omega \lrcorner\,(e^{134} + e^{156}) = e^1, \\ W_5 &= \tfrac{1}{2}\psi_+ \lrcorner\,(e^{1236} + e^{1245}) = e^1. \end{aligned} \tag{8}$$

Now suppose that

$$(d\psi_+)^{3,1} = \Psi \wedge \overline{\sigma} = (\psi_+ + i\psi_-) \wedge \overline{\sigma}.$$

Then $W_5 = 2(\overline{\sigma} + \sigma)$, whereas $\frac{1}{2}\psi_+ \lrcorner\, d\psi_- = 2i(\sigma - \overline{\sigma})$ by (7). It follows that

$$\psi_+ \lrcorner\, d\psi_- = J(\psi_+ \lrcorner\, d\psi_+), \tag{9}$$

a useful re-interpretation of (7).

Each of the components W_1, W_2, W_3 is at worst re-scaled under a conformal change of metric

$$g \mapsto \mathrm{e}^{2f} g. \tag{10}$$

This is a consequence of the fact that none of the corresponding representations in Table 1 is isomorphic to the cotangent space T^* containing the 1-form df. The reduction to $SU(3)$ permits one to define an additional conformally invariant component:

1.3 Proposition The tensor $3W_4 + 2W_5$ is unchanged by (10).

Proof. The transformation (10) multiplies 1-forms by e^f. Hence the exterior derivatives of ω, ψ_+ transform as

$$\begin{aligned} d\omega &\mapsto d(\mathrm{e}^{2f}\omega) = \mathrm{e}^{2f} d\omega + 2\mathrm{e}^{2f} df \wedge \omega, \\ d\psi_+ &\mapsto d(\mathrm{e}^{3f}\psi_+) = \mathrm{e}^{3f} d\psi_+ + 3\mathrm{e}^{3f} df \wedge \psi_+. \end{aligned}$$

Retaining $\lrcorner$ exclusively for the contraction relative to the original metric,

$$\begin{aligned} W_4 &\mapsto W_4 + \omega \lrcorner\,(df \wedge \omega), \\ W_5 &\mapsto W_5 + \tfrac{3}{2}\psi_+ \lrcorner\,(df \wedge \psi_+). \end{aligned}$$

The final terms may be evaluated by using (8) with $df = e^1$, with the result that they cancel out in the sum $3W_4 + 2W_5$. $\square$

2 Static $\mathbf{G_2}$ structures

We denote by $\mathbb{T}$ the space $\mathbb{R}^7$, regarded as the standard representation of the exceptional Lie group G_2. The latter acts transitively on the sphere S^6 in $\mathbb{T}$, and the stabilizer of a point of S^6 is conjugate to a fixed subgroup $SU(3)$ of G_2. The inclusion $SU(3)\subset G_2$ is therefore characterized by the orthogonal decomposition

$$\mathbb{T}=T\oplus\mathbb{R}. \tag{11}$$

We choose an orthonormal basis (e^i) of $\mathbb{T}^*$ such that $\alpha=e^7$ annihilates T at each point.

A G_2-structure on a 7-manifold $\mathbb{M}$ is characterized by a 'positive generic 3-form' φ. Adopting a canonical form compatible with (5), we set

$$\begin{aligned}\varphi&=\omega\wedge\alpha+\psi_+\\&=e^{127}+e^{347}+e^{567}+e^{135}-e^{146}-e^{236}-e^{245}.\end{aligned} \tag{12}$$

The basis (e^i) is orthonormal for the metric determined by the inclusion $G_2\subset SO(7)$, and allows us to consider

$$\begin{aligned}*\varphi&=\psi_-\wedge\alpha+\tfrac12\omega^2\\&=e^{1367}+e^{1457}+e^{2357}-e^{2467}+e^{3456}+e^{1256}+e^{1234}.\end{aligned} \tag{13}$$

The structure of a general G_2-manifold will not reduce to $SU(3)$, and these descriptions are only valid pointwise or locally.

The intrinsic torsion space

$$\mathbb{T}^*\otimes\mathfrak{g}_2^\perp=\mathcal{X}_1\oplus\mathcal{X}_2\oplus\mathcal{X}_3\oplus\mathcal{X}_4$$

has four components of respective dimensions $1,14,27,7$, first described by Fernandez–Gray.[10] Various constructions[1,6,11,12] of metrics with holonomy equal to G_2 are based on the significant fact that the holonomy reduction is characterized by the simultaneous closure of φ and $*\varphi$.

Given that $\mathbb{T}$ represents the tangent space of $\mathbb{M}$, the G_2 counterpart of (2) consists of the decompositions

$$\begin{aligned}\mathbb{T}^*&\cong\mathbb{T}\\ \textstyle\bigwedge^2\mathbb{T}^*&\cong\mathfrak{g}_2\oplus\mathbb{T}\\ \textstyle\bigwedge^3\mathbb{T}^*&\cong\mathbb{R}\oplus\mathbb{T}\oplus S^2_0\mathbb{T}.\end{aligned} \tag{14}$$

It follows that $\mathfrak{g}_2^\perp\cong\mathbb{T}$, and the spaces $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{X}_4$ are isomorphic to $\mathbb{R}$, $\mathfrak{g}_2$, $S^2_0\mathbb{T}$, $\mathbb{T}$ respectively. The components of $\nabla\varphi$ can be recovered from those

Table 2. G_2 torsion

component	dim	G_2-module	$SU(3)$-module			
$\mathcal{X}_1$	1	$\mathbb{R}$	$\mathbb{R}$			
$\mathcal{X}_2$	14	$\mathfrak{g}_2$			T	$\mathfrak{su}(3)$
$\mathcal{X}_3$	27	$S_0^2\mathbb{T}$	$[\![S^{2,0}]\!]$	$\mathbb{R}$	T	$\mathfrak{su}(3)$
$\mathcal{X}_4$	7	$\mathbb{T}$		$\mathbb{R}$	T	

of $d\varphi$ and $d*\varphi$ as follows:

$$
\begin{aligned}
X_1 + X_3 + X_4 &\longleftrightarrow d\varphi,\\
X_1 &\longleftrightarrow d\varphi\wedge\varphi,\\
X_2 + X_4 &\longleftrightarrow d*\varphi,\\
X_4 &\longleftrightarrow (*d*\varphi)\wedge(*\varphi).
\end{aligned}
\tag{15}
$$

The G_2-structure is called *calibrated* (respectively *cocalibrated*) if $d\varphi = 0$ (respectively $d*\varphi = 0$), and examples with all possible combinations of torsion components are known.[13,14,15]

Whilst $\mathcal{X}_4$ is isomorphic to (11), there are analogous decompositions of the other spaces in (14), and therefore of $\mathcal{X}_2$ and $\mathcal{X}_3$, under the action of $SU(3)$. These are described in Table 2. In particular, $S_0^2\mathbb{T}$ also contains a 1-dimensional $SU(3)$-module, and it follows that

$$
\begin{array}{c}
\alpha\\
\omega\\
\psi_+,\quad \omega\wedge\alpha,\quad \psi_-\\
\psi_-\wedge\alpha,\quad \omega^2,\quad \psi_+\wedge\alpha\\
\omega^2\wedge\alpha\\
\omega^3
\end{array}
\tag{16}
$$

is a list of the exterior forms on $\mathbb{M}$ fixed by $SU(3)$. With reference to the summands in (14), we may assert that

2.1 Lemma

$$
\left.\begin{aligned}\psi_-\in\mathbb{T}\\ 3\psi_+-4\omega\wedge\alpha\in S_0^2\mathbb{T}\end{aligned}\right\}\subset \bigwedge{}^3\mathbb{T}^*,\qquad
\left.\begin{aligned}\psi_+\wedge\alpha\in\mathbb{T}\\ 3\psi_-\wedge\alpha-2\omega^2\in S_0^2\mathbb{T}\end{aligned}\right\}\subset \bigwedge{}^4\mathbb{T}^*.
$$

Proof. Given (6), $\gamma = 3\psi_+ - 4\omega \wedge \alpha$ satisfies $\gamma \wedge (*\varphi) = 0$ and is therefore orthogonal to φ. Since $-\psi_-$ is the contraction of $*\varphi$ with the tangent vector dual to α, it lies in the submodule $\mathbb{T}$ of $\bigwedge^3\mathbb{T}^*$. Since γ is also orthogonal to ψ_-, it lies in $S^2_0\mathbb{T}$. The invariant 4-forms are obtained by observing that $*\psi_- = \psi_+ \wedge \alpha$ and $*(\omega \wedge \alpha) = \frac{1}{2}\omega^2$. $\square$

Remark. The existence of various $SU(3)$-invariant elements of $\bigwedge^3\mathbb{T}$ gives rise to a choice of induced G_2-structures in the passage from 6 to 7 dimensions. For example,

$$3\psi_+ - 4\omega \wedge \alpha = 3\left[\omega \wedge (-\tfrac{4}{3}\alpha) + \psi_+\right]$$

determines a G_2 structure with reversed orientation on $\mathbb{T}$ and different scalings relative to (11). Other choices of coefficients will have the effect of modifying combinations in Theorem 3.1 below.

The three components of $\mathbb{T}^* \otimes \mathfrak{g}_2^\perp$ isomorphic to T can be detected from corresponding components of $d\varphi$ and $d{*}\varphi$. It is useful to record the following list for diagnostic purposes.

2.2 Lemma

$$\begin{aligned}
\zeta &= e^{1347} + e^{1567} - e^{1236} - e^{1245} \in T \subset \mathbb{T} \quad \subset {\textstyle\bigwedge}^4\mathbb{T}^*,\\
\eta &= e^{1347} + e^{1567} + e^{1236} + e^{1245} \in T \subset S^2_0\mathbb{T} \subset {\textstyle\bigwedge}^4\mathbb{T}^*,\\
\xi &= \quad e^{13456} + e^{12357} - e^{12467} \quad \in T \subset \mathbb{T} \quad \subset {\textstyle\bigwedge}^5\mathbb{T}^*,\\
\vartheta &= \quad 2e^{13456} - e^{12357} + e^{12467} \quad \in T \subset \mathfrak{g}_2 \quad \subset {\textstyle\bigwedge}^5\mathbb{T}^*.
\end{aligned}$$

Proof. Each of these forms represents the element of T dual to e^1 in an appropriate guise. For example, $\zeta = e^1 \wedge \varphi$ and η is a linear combination of ζ and $\psi_+ \wedge e^1 = e^{1236} + e^{1245}$ orthogonal to ζ. We may define ξ as $e^1 \wedge (*\varphi) = *(e^1 \lrcorner\, \varphi)$. Then ϑ is a linear combination of ξ and $e^1 \wedge \omega^2 = 2e^{13456}$ orthogonal to ξ. $\square$

3 Product manifolds

We now suppose that $\mathbb{M}$ is a 7-manifold with an $SU(3)$-structure, so that the differential forms $\alpha, \omega, \psi_+, \psi_-$ of respective degrees $1, 2, 3, 3$ and constant norm are all defined globally. In this and the following sections, we shall investigate properties of the G_2-structure defined with the convention of (12). In general, one may write $d\alpha = \alpha \wedge \beta + \gamma$, where β, γ are forms with values in the subspace T^* at each point. For example, the equation $\gamma = 0$ is the integrability condition for the 6-dimensional distribution now determined by (11). We shall consider various special cases, the simplest of which is that in which $\mathbb{M}$ is the Riemannian product of M with an interval or circle, so that $\nabla\alpha$ (and so $d\alpha$) vanishes.

In the product situation, we choose to write $\alpha = e^7 = dt$, so that

$$\begin{aligned} d\varphi &= d\omega \wedge dt + d\psi_+, \\ d*\varphi &= d\psi_- \wedge dt + \omega \wedge d\omega. \end{aligned} \tag{17}$$

Let

$$\begin{aligned} \tau_1 &= (W_1^+ + W_1^-) + (W_2^+ + W_2^-) + W_3 + W_4 + W_5, \\ \tau_2 &= X_1 + X_2 + X_3 + X_4 \end{aligned}$$

denote the respective intrinsic torsion tensors, as defined in the previous sections. Since $\nabla\varphi$ can be computed in terms of $\nabla\omega$ and $\nabla\psi_+$, the tensor τ_2 is determined by τ_1. Any $SU(3)$-invariant component of τ_2 must be a linear combination of W_1^+ and W_1^-, and any component isomorphic to T a linear combination of W_4 and W_5. The precise statement is

3.1 Theorem The four components of τ_2 are determined by the seven components of τ_1 as follows.

$$\begin{aligned} X_1 &\longleftrightarrow W_1^+ \\ X_2 &\longleftrightarrow (W_2^-,\ 2W_4 + W_5) \\ X_3 &\longleftrightarrow (W_1^+,\ W_2^+,\ W_3,\ W_4 + W_5) \\ X_4 &\longleftrightarrow (W_4 - W_5,\ W_1^-). \end{aligned}$$

Proof. The component X_1 arises from

$$\varphi \wedge d\varphi = \psi_+ \wedge d\omega \wedge dt + \omega \wedge dt \wedge d\psi_+ = 2W_1^+ \omega^3 \wedge \alpha.$$

Similarly, X_4 is determined by the contraction

$$\varphi \lrcorner\, d\varphi = -\omega \lrcorner\, d\omega + \psi_+ \lrcorner\, d\psi_+ + (\psi_+ \lrcorner\, d\omega)dt.$$

The first two terms of the right-hand side are $-2W_4$ and $2W_5$ by Definition 1.2, and the last term corresponds to $d\omega \wedge \psi_-$ (given that $\psi_+ \wedge \psi_+ = 0$) and so W_1^-. This justifies the description of X_4.

The hypotheses $d\omega = \omega \wedge e^1$ and $d\psi_- = \psi_- \wedge e^1$ are compatible with the constraint $W_4 = W_5$. In this case

$$d*\varphi = -e^{12357} + e^{12467} + 2e^{13456} = \vartheta,$$

confirming that $X_4=0$. In order to obtain ξ instead of ϑ, we need to take $d\omega=\frac{1}{2}\omega\wedge e^1$ and $d\psi_-=-\psi_-\wedge e^1$, which corresponds to $2W_4+W_5=0$.

The association of W_2^- with X_2 and W_2^+,W_3 with X_3 follows immediately from (15). The hypotheses $d\omega=\omega\wedge e^1$ and $d\psi_+=-\psi_+\wedge e^1$ are compatible with the constraint $W_4+W_5=0$. This implies that

$$d\varphi=e^{1347}+e^{1567}-e^{1236}-e^{1245}=\zeta,$$

whence the T-component of X_3 is proportional to W_4+W_5. $\square$

Remark. The difference $49-42=7$ of the dimensions of the spaces containing τ_1 and τ_2 is accounted for by the repetition of W_1^+ and a linear combination of W_4,W_5 in the above list. This redundancy is eliminated in the more complicated situations described in subsequent sections. The lack of repetiton between components of X_1,X_2 is consistent with the result[14] that a connected G_2 manifold with $\tau_2\in\mathcal{X}_1\oplus\mathcal{X}_2$ has at least one of X_1,X_2 zero.

3.2 Corollary Suppose that M has an $SU(3)$-structure. The G_2-structure defined on $M\times\mathbb{R}$ by (12) is cocalibrated if and only if $\tau_1\in\mathcal{W}_2^-$.

Examples. 1. An almost-Hermitian 6-manifold is called nearly-Kähler[16] if ∇J belongs to the space $\mathcal{W}_1$. Assuming that $\nabla J\neq 0$, the structure reduces to $SU(3)$ and we may suppose that $\tau_1\in\mathcal{W}_1^-$ with ψ_+ is proportional to $d\omega$. The product $M\times S^1$ then has a G_2-structure with $\tau_2\in\mathcal{X}_4$. Alternatively we may swap the roles of ψ_+,ψ_- to obtain $\tau_2\in\mathcal{X}_1\oplus\mathcal{X}_3$.

2. A known example[13] of a calibrated nilmanifold can be interpreted as follows. Let $\mathfrak{g}=\mathbb{R}\oplus\mathfrak{h}$ be a 6-dimensional Lie algebra with structure determined by

$$de^i=\begin{cases}0, & i=1,2,4,5\\ e^{25}, & i=3,\\ -e^{24}, & i=6.\end{cases}$$

The definitions (5) furnish an associated nilmanifold $M=\Gamma\backslash G$ with an $SU(3)$-structure for which

$$d\omega=0,\quad d\psi_+=0,\quad d\psi_-=e^{1234}-e^{1256},$$

whence $\tau_1\in\mathcal{W}_2^-$. It follows that $M\times S^1$ has both a calibrated G_2 structure and (swapping ψ_+,ψ_-) a cocalibrated one with $\tau_2\in\mathcal{X}_3$. The same Lie algebra $\mathfrak{g}$ was incidentally used[17] in the construction of a closed non-parallel 4-form with stabilizer $Sp(2)Sp(1)$.

4 Dynamic $\mathbf{G_2}$ structures

Let M be a fixed 6-manifold. Suppose that (ω, ψ_+, ψ_-) is an $SU(3)$ structure that depends on a real parameter t lying in some interval $I \subseteq \mathbb{R}$, so that one may regard $\mathbb{M} = M \times I$ as a warped product fibring over I.

To avoid confusion, we denote exterior differentation on M by $\hat{d}$ in this section. Adopting a unit 1-form dt on I allows us to write

$$\begin{aligned} d\varphi &= (\hat{d}\omega - \frac{\partial \psi_+}{\partial t}) \wedge dt + \hat{d}\psi_+, \\ d * \varphi &= (\hat{d}\psi_- + \omega \wedge \frac{\partial \omega}{\partial t}) \wedge dt + \omega \wedge \hat{d}\omega. \end{aligned}$$

This motivates

4.1 Definition An almost Hermitian 6-manifold is *half-flat* if it possesses a a reduction to $SU(3)$ for which $d\psi_+ = 0$ and $\omega \wedge d\omega = 0$.

Half-flatness is therefore characterized by the closure of ψ_+ and ω^2. It amounts to requiring that $W_1 + W_2$ has rank one and that both W_4, W_5 vanish. This eliminates $1 + 8 + 6 + 6 = 21$ of the total 42 dimensions of τ_1, which is constrained to lie in $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$.

If we now suppose that the G_2-structure on $\mathbb{M}$ has holonomy group contained in G_2, we may conclude that M is half-flat for all t, and that the forms evolve according to the equations

$$\begin{cases} \hat{d}\omega = \dfrac{\partial \psi_+}{\partial t}, \\ \hat{d}\psi_- = -\omega \wedge \dfrac{\partial \omega}{\partial t}. \end{cases} \tag{18}$$

Conversely, suppose we are given a half-flat $SU(3)$-structure at time $t = t_0$. The equations (18) may be regarded as a system constraining a closed 3-form ψ_+ and a closed 4-form ω^2. For, if (as in our situation) the stabilizer of ψ_+ is $SL(3, \mathbb{C})$ then ψ_+ determines ψ_- via (3). In this way, Hitchin[4] proved that the compatibility equations (6) are conserved in time, and this leads to

4.2 Theorem Let M be an almost Hermitian 6-manifold which is half-flat. Then there exists a metric with holonomy contained in G_2 on $M \times I$ for some interval I.

A key example of this construction is the following. Given $(M, \hat{g})$, consider the conical metric $g = t^2\hat{g} + dt^2$ on $M \times \mathbb{R}^+$. Consistent with this, we set

$$\omega = t^2\hat{\omega}, \quad \psi_+ = t^3\hat{\psi}_+, \quad \psi_- = t^3\hat{\psi}_-,$$

where circumflex indicates a form independent of t. Then (18) becomes

$$d\hat{\omega} = 3\hat{\psi}_+, \quad d\hat{\psi}_- = -2\hat{\omega}^2.$$

These equations determine the nearly-Kähler class for which $\tau_1 \in \mathcal{W}_1^-$.

Further examples. 1. We now construct metrics with holonomy G_2 associated to each of the three nilmanifolds with $b_1 = 4$ whose $U(3)$ structures were classified by Abbena et al.[18] Starting with the Iwasawa manifold M, we modify the usual basis of 1-forms in order that

$$de^i = \begin{cases} 0, & i = 1,2,3,4, \\ -e^{14} - e^{23}, & i = 5, \\ -e^{13} - e^{42}, & i = 6. \end{cases} \tag{19}$$

The metric

$$g = t^2 \sum_{i=1}^{4} e^i \otimes e^i + t^{-2} \sum_{i=5}^{6} e^i \otimes e^i + t^4 dt^2$$

is compatible with the natural fibration $M \to T^4$, and the forms

$$\begin{aligned} \omega &= t^2(e^{12} + e^{34}) + t^{-2}e^{56}, \\ \psi_+ + i\psi_- &= t(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \end{aligned}$$

determine a reduction to $SU(3)$ for which

$$\hat{d}\omega = t^{-3}\psi_+, \quad \hat{d}\psi_- = -4t\, e^{1234},$$

whence $\tau_1 \in \mathcal{W}_1^- \oplus \mathcal{W}_2^-$. Indeed, g is a metric with holonomy G_2 on $M \times \mathbb{R}^+$.

2. The previous example was discovered by Gibbons et al[5] who exhibit it as arising from the standard complete metric with holonomy G_2 by 'contracting' the isometry group $SO(5)$. A more complicated 2-step example is associated to the Lie algebra with

$$de^i = \begin{cases} 0, & i = 1,2,3,4, \\ -e^{14} - e^{23}, & i = 5, \\ e^{24}, & i = 6. \end{cases}$$

Consider an orthonormal basis of 1-forms

$$te^1, \quad t^2e^2, \quad te^3, \quad t^2e^4, \quad t^{-2}e^5, \quad t^{-1}e^6, \quad 2t^4dt,$$

and the $SU(3)$-structure for which

$$\omega = t^3(e^{12}+e^{34}) + t^{-3}e^{56},$$
$$\psi_+ + i\psi_- = (e^1+ite^2)\wedge(e^3+ite^4)\wedge(e^5+ite^6),$$

by analogy to (5). This yields a G_2-structure with closed forms

$$\varphi = 2t^7(e^{12}+e^{34})\wedge dt + 2te^{56}\wedge dt + e^{135} - t^2(e^{146}+e^{236}+e^{245}),$$
$$*\varphi = -2t^7e^{246}\wedge dt + 2t^5(e^{145}+e^{136}+e^{235})\wedge dt + e^{1256}+e^{3456}+t^6e^{1234}.$$

3. The nilpotent 3-step Lie algebra for which

$$de^i = \begin{cases} 0, & i=1,2,4,5 \\ e^{25}, & i=3, \\ e^{14}-e^{23}, & i=6 \end{cases}$$

gives rise to an example satisfying the hypotheses of Theorem 4.2. In fact, the forms (5) satisfy

$$\omega\wedge d\omega = 0, \quad d\psi_+ = 0, \quad d\psi_- = -e^{1256}.$$

An explicit determination of the metric requires an analysis of spaces of invariant exact forms.[4]

5 Fibred $\mathbf{G_2}$ manifolds

Let $(M,\hat{g})$ be a Riemannian 6-manifold. In this final section, we consider the 7-dimensional total space of a circle fibration $\pi:\mathbb{M}\to M$, endowed with a metric of the form

$$g = \alpha\otimes\alpha + \pi^*\hat{g}, \tag{20}$$

with $d\alpha = \pi^*\rho$ for some 2-form ρ on M.

To begin with, suppose that M has an $SU(3)$-structure. The G_2-structure on $\mathbb{M}$ determined by (12) satisfies the following enhanced version of (17), in which we omit the pullback operator π^*:

$$\begin{aligned} d\varphi &= d\omega\wedge\alpha + d\psi_+ + \omega\wedge\rho. \\ d*\varphi &= d\psi_-\wedge\alpha + \omega\wedge d\omega - \psi_-\wedge\rho. \end{aligned} \tag{21}$$

Indicating the components of the 2-form ρ by

$$\begin{array}{cccccc} \bigwedge^2 T^* = & \langle\omega\rangle & \oplus & [\Lambda_0^{1,1}] & \oplus & [\![\Lambda^{2,0}]\!] \\ \rho \quad = & \rho_0\omega & + & \rho_1 & + & \rho_2, \end{array}$$

we obtain the following generalization of Theorem 3.1.

5.1 Theorem The four components of τ_2 are given by

$$\begin{aligned}
X_1 &\longleftrightarrow W_1^+ + \rho_0\\
X_2 &\longleftrightarrow (W_2^-,\ 2W_4{+}W_5{-}2\rho_2)\\
X_3 &\longleftrightarrow (3W_1^+ - 4\rho_0,\ W_2^+{+}\rho_1,\ W_3,\ W_4{+}W_5{+}\rho_2)\\
X_4 &\longleftrightarrow (W_4{-}W_5{-}\rho_2,\ W_1^-).
\end{aligned}$$

Proof. Whilst X_1 corresponds to $\varphi\wedge d\varphi$, Lemma 2.1 tells us that the 1-dimensional component in X_3 arises from $(3\omega\wedge\alpha-4\psi_+)\wedge d\varphi$. The various coefficients of ρ_2 can be deduced from the observations: (i) $d\varphi=0$ implies $d\omega=0$, and (ii) $d*\varphi=0$ implies $d\psi_-=0$. □

A simple case is that in which the entire torsion τ_1 vanishes, which corresponds to

$$\begin{aligned}
d\varphi &= \omega\wedge\rho,\\
d*\varphi &= -\psi_-\wedge\rho = -\psi_-\wedge\rho_2.
\end{aligned}$$

This situation applies when M is the torus $T^6=\mathbb{R}^6/\mathbb{Z}^6$ endowed with a constant $SU(3)$-structure. The set (1) of such structures compatible with $\hat g$ is isomorphic to the set of G_2-structures compatible with (20) (assuming that orientations are also preserved). We deduce that $\mathbb{M}$ is calibrated if and only if $\rho=0$, and cocalibrated if and only if $\rho_2=0$.

Examples. 1. Let $\mathbb{M}$ be a circle bundle with curvature 2-form

$$\rho = e^{12}+e^{34}+e^{56}$$

over T^6. Then there are no calibrated G_2 structures on $\mathbb{M}$ compatible with g. The set $\mathcal{C}^*$ of cocalibrated structures corresponds to $SU(3)$-structures relative to which ρ has type $(1,1)$. By considering first the space $\mathbb{CP}^3$ (regarded as the set $SO(6)/U(3)$ of orthogonal almost complex structures[18,19]), $\mathcal{C}^*$ can be seen to be a disjoint union $\mathbb{RP}^1\sqcup\mathbb{RP}^5$.

2. We can apply the above theory to the real projective space $\mathbb{RP}^7$ by representing it as $SO(5)/SU(2)$, which fibres over $SO(5)/U(2)\cong\mathbb{CP}^3$. It is well known that the latter has a homogeneous nearly-Kähler metric, so that there is a reduction to $SU(3)$ with $\tau_1\in\mathcal{W}_1^\pm$. Moreover, the curvature 2-form ρ of the above Hopf fibration is an element of $[\Lambda_0^{1,1}]$ relative to the non-integrable complex structure.[20] Example 1 of Section 3 now implies that $\mathbb{RP}^7$ has a G_2-structure with $\tau_2\in\mathcal{X}_1\oplus\mathcal{X}_3$ and another with $\tau_2\in\mathcal{X}_3\oplus\mathcal{X}_4$.

3. Let us return to the descriptions (1). The space of G_2-invariant differential forms on $\mathbb{RP}^7$ is 1-dimensional, and it follows that $\mathbb{RP}^7$ has a nearly-parallel G_2-structure, one with $\tau_2 \in \mathcal{X}_1$. More generally, we may regard (16) as a list of $SO(6)$-invariant differential forms on $\mathbb{RP}^7$ subject to the relations $d\alpha = -\frac{1}{2}\omega$ and

$$d\psi_\pm = \alpha \wedge \psi_\mp, \tag{22}$$

that also give rise to G_2-structures with $\tau_2 \in \mathcal{X}_1 \oplus \mathcal{X}_3$.

One can arrive at (22) by considering the canonical S^1-bundle $\mathbb{B}$ over a Kähler manifold M of real dimension 6 (see (4)). A local orthonormal basis $\{\psi^1, \psi^2\}$ of sections of the bundle with fibre $[\![\Lambda^{3,0}]\!] \cong \mathbb{R}^2$ gives rise to coordinates a^1, a^2 and radial parameter $r = \sqrt{(a^1)^2 + (a^2)^2}$ on $\mathbb{B}$. The Kähler condition implies that $d\psi^1 = \sigma \wedge \psi^2$ for some 1-form σ on M, and we use

$$b^1 = da^1 - a^2\pi^*\sigma, \quad b^2 = da^2 + a^1\pi^*\sigma$$

to define global forms

$$\begin{aligned} rdr &= a^1b^1 + a^2b^2, & \alpha &= a^1b^2 - a^2b^1, \\ \psi_+ &= a^1\psi^1 + a^2\psi^2, & \psi_- &= a^1\psi^2 - a^2\psi^1. \end{aligned}$$

A straightforward calculation[2] shows that

$$rdr \wedge \psi_+ + \alpha \wedge \psi_- = r^2(b^1 \wedge \psi^1 + b^2 \wedge \psi^2) = r^2 d\psi_+,$$

so that restricting to $\mathbb{B}$ $(r = 1)$ we obtain (22). Similarly, $d\alpha = \pi^*\rho$, where $\rho = d\sigma$ can be identified with the Ricci form. This leads to the following result of Baum et al,[21] which forms part of a more general theory of nearly-parallel G_2 structures.[22,15]

5.2 Theorem If M is Kähler-Einstein with positive scalar curvature then $\mathbb{B}$ has a G_2-structure with $\tau_2 \in \mathcal{X}_1$.

We conclude this article by returning to the initial set-up of this section, together with the assumption that the holonomy of the metric (20) reduces to the subgroup G_2 determined by the 3-form (12). This allows us to set $X_i = 0$ in Theorem 5.1.

5.3 Corollary If the holonomy group of the metric (20) reduces to G_2 then the quotient $M = \mathbb{M}/S^1$ has an $SU(3)$-structure for which $\tau_1 \in \mathcal{W}_2^+$.

Observe that the resulting condition on τ_1 involves a change of sign from that in Corollary 3.2. Indeed, from (21), we obtain $d\omega = 0$ and $d\psi_- = 0$. It is convenient to regard $SU(3)$-structures with $(d\psi_-)^{2,2} = 0$ as 'self-dual', and those with $(d\psi_+)^{2,2} = 0$ as 'anti-self-dual'. The latter type occurred naturally in Sections 3 and 4, and we focus attention on the $(2,2)$ components so that the terminology is conformally invariant. In the present situation, we may therefore say that M has a self-dual symplectic structure with

$$d\psi_+ = \omega \wedge \rho,$$

where $\rho = \rho_1$ is a closed effective $(1,1)$-form.

Whilst the incomplete examples of Section 4 admit quotients of this type, a more realistic generalization of the condition $\tau_1 = 0$ is obtained by dropping the assumption that the S^1 orbits on $\mathbb{M}$ have constant size. In this situation, there exists a function f on M for which $d(\mathrm{e}^{2f}\omega) = 0$. It follows that, applying the conformal transformation (10), M once again has a self-dual symplectic structure, though this time $W_5 = -df$ is non-zero and

$$\rho = \rho_1 + \rho_2 = -e^{-3f}(W_2^+ + 2df \lrcorner \psi_-)$$

is closed. A study of this particular class of structures may be valuable in the construction of metrics with holonomy G_2.

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